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The Near Horizon Geometry of a Black Hole

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Introduction

Black holes are regions of space where mass is compacted so densely, and has gravity so strong, that not even light can escape. Originally, when they were discovered as a consequence of Einstein's General Relativity, they were deemed a mathematical curiosity with no physical significance. This is no longer the consensus, as we have since observed their effects on the motion of celestial bodies, the gravitational waves they produce, and the accretion disks of high-temperature matter that often surround them. Since discovering that black holes exist in our universe, there have been other apparent problems with their description that have been overcome through careful consideration. For our research this past year, we have used the geometric properties of black holes and quantum mechanics to explore interesting aspects about the nature of black holes.

The Schwarzschild Metric

The metric is a matrix that we use to define the spacetime interval (the analogue to distance) in relativity. In flat spacetime, we use the *Minkowski Metric* which produces the interval:

$$ds^2 = d\tau^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

For curved spacetime, Karl Schwarzschild discovered a new metric, known as the *Schwarzschild Metric*, that obeys Einstein's field equations and manifests itself in a geometry that is both spherically symmetric and static. This new metric produces an interval

$$ds^2 = d\tau^2 = -\left(1 - \frac{2MG}{r}\right)dt^2 + \left(1 - \frac{2MG}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 sin^2\theta^2 d\phi^2$$

where ***r*** and ***t*** are coordinate radius and time respectively. These are the quantities that an observer far away from the black hole would measure. Due to the spherically symmetric curvature of spacetime, the measured radial distance becomes stretched compared to the coordinate distance. Another important feature of this new metric is that it diverges as the radius approaches *2MG*. We call this distance the *Schwarzschild Radius (Rs),* or for black holes, the *Event Horizon.* Normally, when we see divergence in our math it means that something has gone terribly wrong. However, using some clever coordinate transformations we can get around this problem.

Uniform Acceleration in Minkowski Space

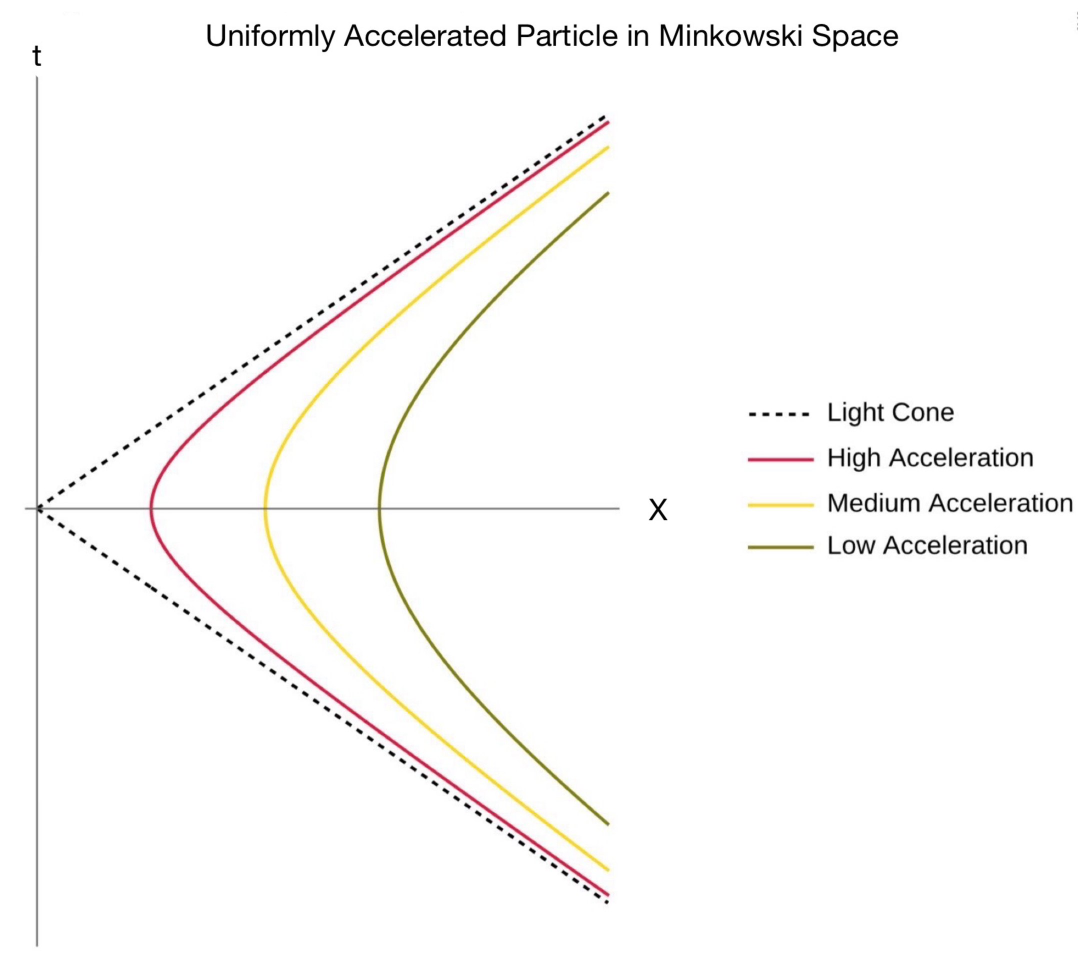
In Minkowski space, which is flat spacetime, an object that experiences constant acceleration will take a hyperbolic path, of the form

$$x^2 - t^2 = \frac{1}{a^2}$$

through spacetime as shown in the diagram below. In a non-relativistic analysis, this trajectory would be parabolic as simply integrating **a** twice produces

$$x(t) = \frac{a}{2}t^2 \text{ when } x_0 = v_0 = 0$$

However, because relativity limits the maximum velocity to the speed of light, the velocity of a uniformly accelerating object will asymptotically approach C, producing a hyperbolic trajectory like the ones plotted below.



Rindler Coordinates

Due to the hyperbolic path undertaken by the uniformly accelerated observer in Minkowski space, we now introduce the *Rindler metric* to simplify our near horizon study. This coordinate system assumes a constantly accelerated frame and the Rindler interval near the horizon is given by

$$ds^2 = d\tau^2 = -\rho^2 d\omega^2 + d\rho^2 + dx^2 + dy^2$$

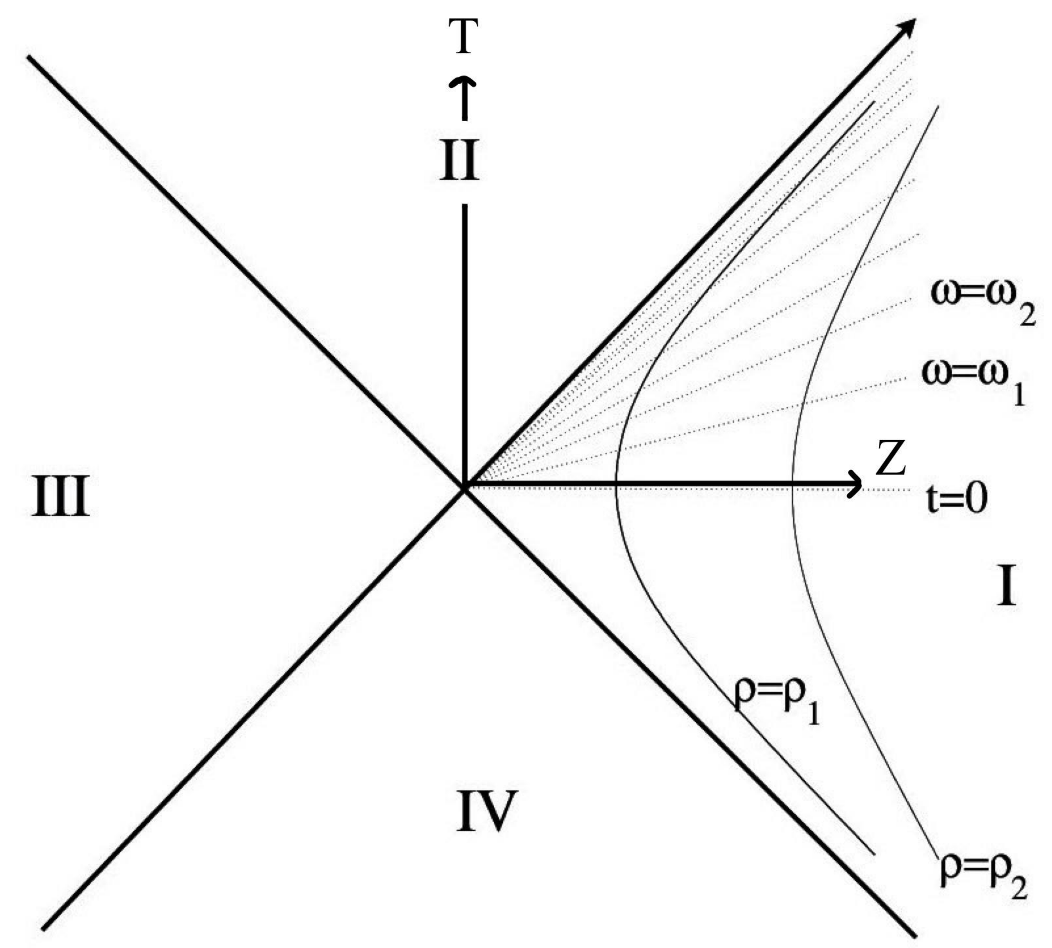
$$\text{where } \omega = \frac{t}{4MG}, x = 2MG\theta cos\phi, y = 2MG\theta sin\phi$$

and **r** has been replaced by **ρ**, or the proper distance from the horizon. Additionally, the metric relies on the use of a dimensionless time **ω**. We can easily arrive back to our familiar Minkowski metric if we limit ourselves to **r** near *2MG*, and a small angular region:

$$ds^2 = d\tau^2 = -dT^2 + dZ^2 + dX^2 + dY^2$$

$$\text{where } T = \rho\, sinh\,\omega, Z = \rho\, cosh\,\omega$$

Overall, this coordinate transformation informs us that the event horizon (*Rs = 2MG)* is locally nonsingular and the space around *Rs* is locally flat. Spacetime diagrams, like the one to the right, allow us to visually understand this metric. Here, the event horizon is the origin, and the four quadrants represent regions separated by it. Region **I** is located outside of the horizon and in addition to representing Rindler space, it is causally disconnected from regions **III** and **IV**. Importantly, region **II** is causally disconnected from all regions, showing that nothing can ever leave from inside the event horizon. Moreover, we observe that the light cones present in our Minkowski diagram for an accelerated observer are preserved and increasing values of **ρ** correspond to decreasing values of acceleration for our accelerated observer in flat spacetime.



From *An Introduction to Black Holes* by Susskind

Unruh and Hawking Radiation

In Quantum Field Theory, when an observer in flat space and an inertial frame measures their vacuum energy by acting on the vacuum state $|0_f\rangle$, with the *number operator* $\mathbf{n}_i$, they measure 0. However, if this observer were to be in a curved space – or by the equivalence principle, accelerating in flat space – this would no longer be the case as the quantum field being measured is defined differently between flat and curved space. Using the *Bogoliubov Transformation*, we can relate the fields in flat and curved space and subsequently determine how the number operator in curved space acts on the flat space. We find that

$$\langle 0_f | \mathbf{n}_{gi} | 0_f \rangle = \sum_k \beta^*_{ik}\beta_{ik} \neq 0$$

which means an observer in curved space measures a non-zero vacuum energy in flat space. We call this the *Unruh Effect*, and the thermal radiation it creates *Unruh Radiation* – or in the special case of black holes, *Hawking Radiation*. The *Unruh Temperature* and *Hawking Temperature* measure the temperature of their radiation and are

$$T_U = \frac{a}{2\pi} \;\&\; T_{BH} = \frac{1}{8\pi MG}$$

respectively.

Outlook and Conclusion

The Black Hole Information Paradox, originally proposed by Hawking in a 1976 paper, arises from a fundamental disagreement between the quantum mechanical and relativistic predictions of how black holes thermalize (scramble and re-emit as radiation) information. In trying to answer this question the community stands to learn a lot about the true nature of gravity, and how our two most complete descriptions of the universe fit and do not fit together. Our work this past year has covered invaluable prerequisite knowledge on general relativity and quantum mechanics in order to equip us with the tools to tackle this problem.

References: L. Susskind and J. Lindesay. *An Introduction to Black Holes, Information and the String Theory Revolution: The Holographic Universe*. G-Reference,Information and Interdisciplinary Subjects Series. World Scientific, 2005.
S. W. Hawking. "Breakdown of predictability in gravitational collapse". Physical review. D, Particles and fields 14.10 (1976), pp. 2460–2473. issn: 0556-2821.